Math 4200-001
Week 1 concepts and homework
1.1-1.3
Due Wednesday September 2, at start of class.

1) On the history of complex numbers: There is what looks to be a very nice series of videos for a complex analysis course which intersects our own. The class was created by Dr. Petra Bonfert-Taylor and the first lecture talks about the origins - one could work formally with a symbol whose square was -1, in order to find at least on real root to every cubic equation. Nothing to hand in here, but you might decide you want to watch some more of those videos as a supplement to our class at appropriate times.
https://www.youtube.com/c/PetraBonfertTaylor/playlists

"Worked examples" are before the problem set, have

Text problems: There are example exercises at the end of each section which I recommend looking over before you try the homework. Odd answers are in the back of the text.

because the HW ideas are often similar

1.1 1b, 2c, 4ab, 6a, 10, 11 (just check the distributive law), 14, 17a.

1.2 1a, 2b, 4, 5, 8, 11, 14, 19.

1.3 1a, 4b, 5a, 6a, 7a, 8a, 10, 21, 23, 30b.

do these! & turn them in.

w1.1 Sketch the following subsets of the complex plane. Use shading and labeling to clearly specify the subset.
a) $\{z \in \mathbb{C} \mid 1 \leq \mathrm{Re}(z) < 3\,, 0 < \mathrm{Im}(z) < 2\}$.
b) $\left\{z \in \mathbb{C} \,\middle|\, |z| \leq 2, 0 < arg(z) < \frac{\pi}{2}\right\}$
c) The image of the sector in b), under the transformation $f(z) = z^3$.

w1.2 Sketch the following subsets of the complex plane, as above.
a) $\left\{z \in \mathbb{C} \,\middle|\, -\frac{\pi}{4} \leq \mathrm{Im}(z) \leq \frac{\pi}{4}\right\}$
b) The image of the strip in a), under the transformation $f(z) = e^z$.
c) The image of the right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ under the transformation $g(z) = \log(z)$, where $arg(1)$ is chosen to be 2π.

Math 4200-001
Monday August 31: 1 Section 1.3, complex transformations.

Announcements

HW due Wed, 'til 5:00 p.m on Gradescope
(all problems ~ but only some will be graded.)

On Friday we showed that every non-zero complex number has two square roots. I meant to tell you right after that, and before we talked about the general fundamental theorem of algebra, that a consequence of the square root analysis is that the quadratic formula for any degree 2 quadratic equation

$$a\,\boldsymbol{z}^2 + b\,\boldsymbol{z} + c = 0,\ a,\ b,\ c \in \mathbb{C},\ a \neq 0$$
$$\boldsymbol{z} = -\frac{b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

always yields two roots, counting multiplicity. (The $\pm\sqrt{b^2 - 4\,a\,c}$ represents the two square roots of the discriminant, when it's non-zero.)

Warm-up exercise

On Friday we began discussing complex transformations f from $\mathbb{C} \rightarrow \mathbb{C}$. Using polar form we saw that the affine transformations

Example 1

$$f(\boldsymbol{z}) = a\,\boldsymbol{z} + b$$

are compositions of (1) a rotation, followed by (2) a scaling, followed by (3) a translation: Writing

$$\boldsymbol{z} = |\,\boldsymbol{z}\,|\,\mathrm{e}^{i\,arg(z)},$$
$$a = |a\,|\,\mathrm{e}^{i\,arg(a)},$$

we wrote

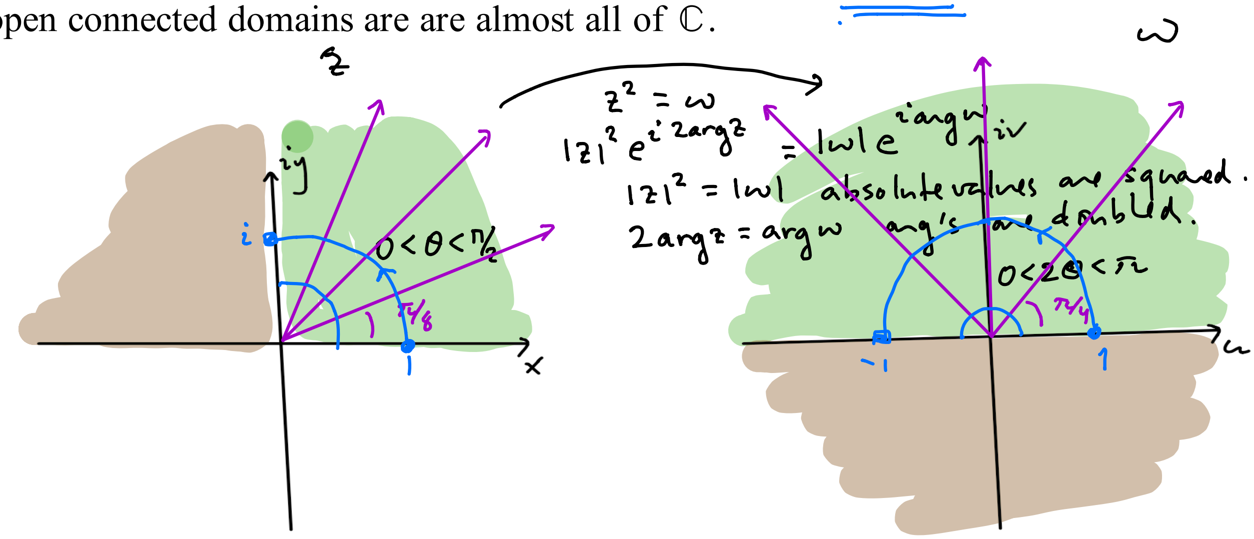
$$f(\boldsymbol{z}) = |a\,|\mathrm{e}^{i\,arg\,a}\,\boldsymbol{z} + b =$$
$$f = f_3 \circ f_2 \circ f_1$$
$$f_1(z) = \mathrm{e}^{i\,arg(a)}\boldsymbol{z} \qquad \text{rotate}$$
$$f_2(w) = |a|\,w \qquad \text{scale}$$
$$f_3(q) = q + b \qquad \text{translate.}$$

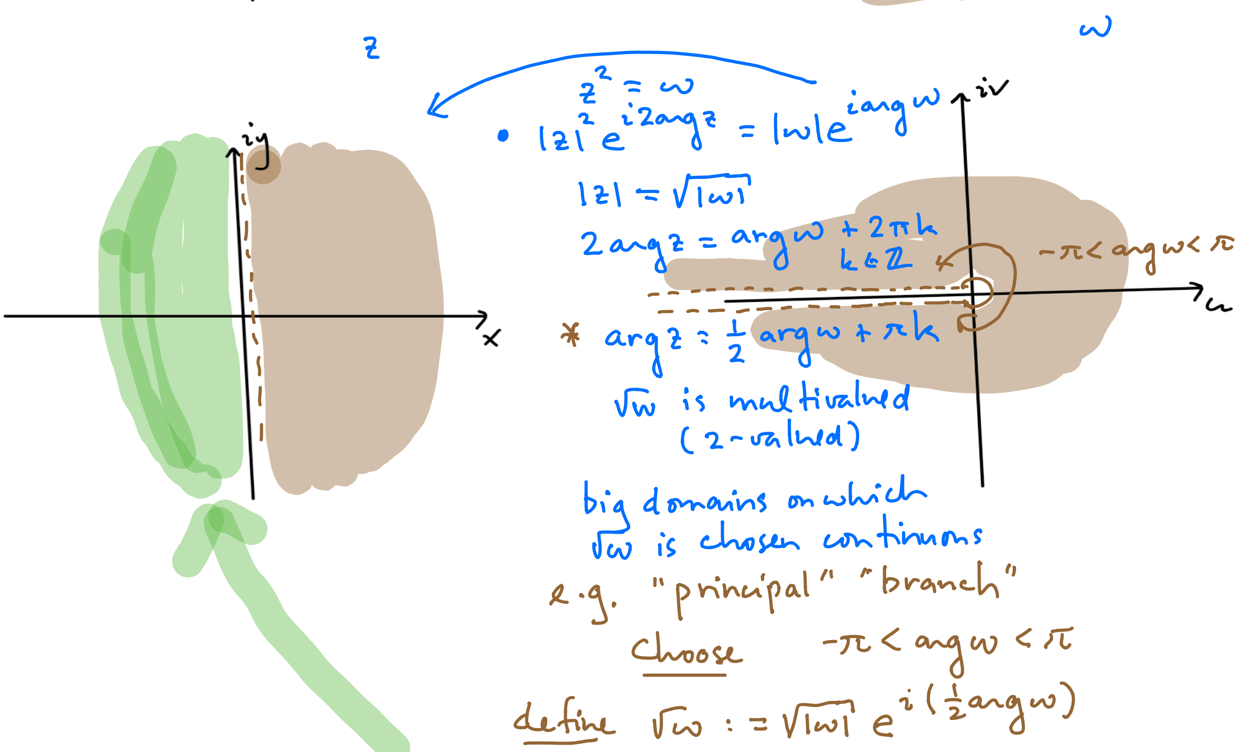
And we saw that it was possible to describe the corresponding transformation $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and the equivalent decomposition using real-variables rotations, scalings, and translations.

Example 2

- $f(z) = z^2$
- $z = |z|\, e^{i\, arg(z)}$
- $f(z) = |z|^2\, e^{i\, 2\, arg(z)}$

Discuss and sketch how f transforms z − plane into a (mostly) twice-covered w − plane. For $w = z^2$ discuss possible continuous inverse functions $z = \sqrt{w}$, and corresponding open connected domains are are almost all of $\mathbb{C}$.





big domains on which $\sqrt{w}$ is chosen continuous

e.g. "principal" "branch"

Choose $-\pi < \arg w < \pi$

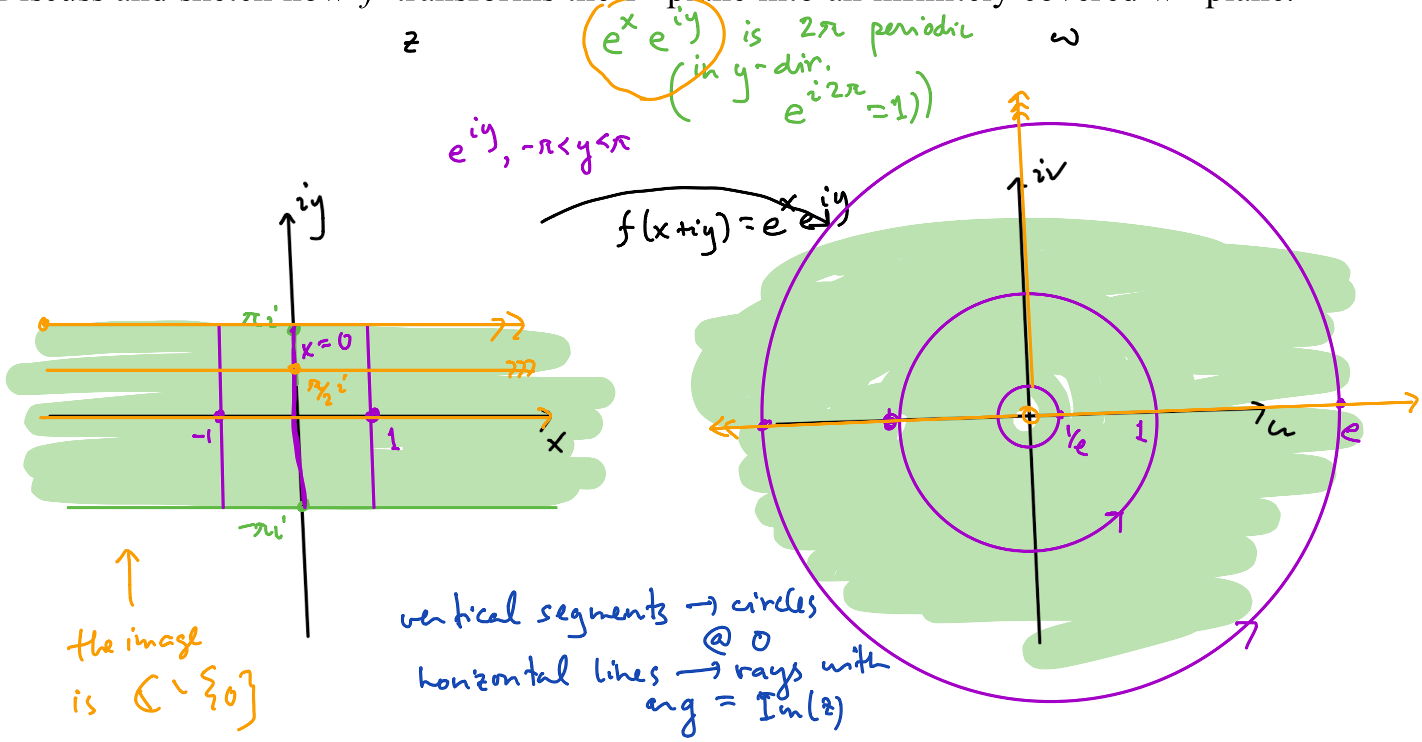
define $\sqrt{w} := \sqrt{|w|}\, e^{i(\frac{1}{2}\arg w)}$

another branch would've use same w domain but added π to arguments

Example 3 For $z = x + i\,y$, $x, y \in \mathbb{R}$

$$f(z) = e^z = e^{x + i\,y} := e^x\, e^{i\,y}$$

Discuss and sketch how f transforms the z-plane into an infinitely covered w-plane.



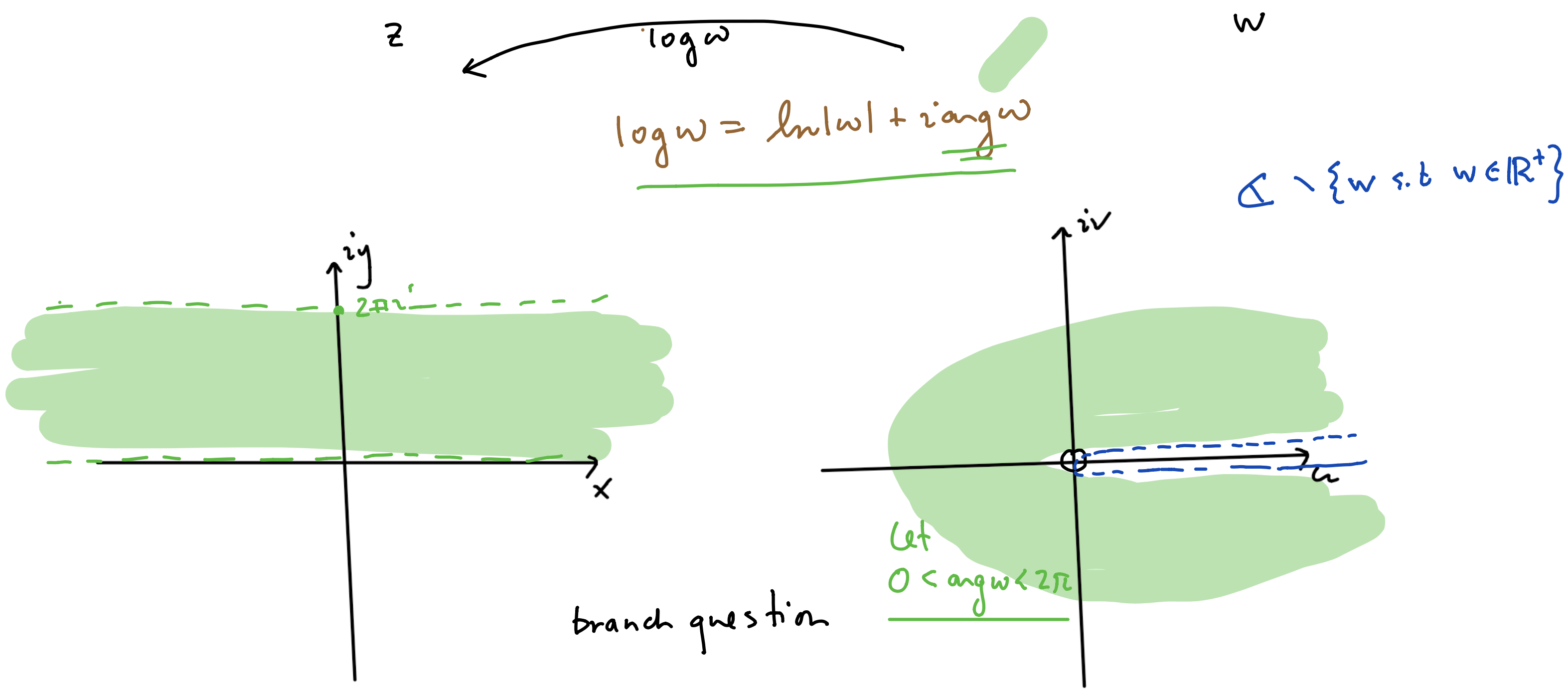
Example 4 For $w = e^{z}$ use polar form to find the multi-valued inverse "function" $z = \log(w)$, and corresponding domains:

$$e^{z} = e^{x + i y} = w = |w|\, e^{i\, arg(w)}$$

$$e^{x} e^{iy} = |w| e^{i \arg w}$$

$$|\ |: \quad e^{x} = |w| \Rightarrow x = \ln|w| \qquad (w \neq 0)$$

$$\arg: \quad y = \arg w + 2\pi k$$



Remark: The logarithm is used to define complex powers of complex numbers, in analogy with the definition in real variables. It is not too hard to check that this definition generalizes the integer powers and roots that we've already talked about.

Definition:

$$w^{z} := e^{z \log w}.$$

(I assigned a few hw problems which are examples of this definition.)

e.g. i^{i} see hw

Example 5 "Trig functions".
If x is real,

$$\text{eqtn 1} \qquad e^{ix} = \cos(x) + i\sin(x)$$

$$\text{eqtn 2} \qquad e^{-ix} = \cos(x) - i\sin(x).$$

$\dfrac{eqtn\ 1 + eqtn2}{2} \Rightarrow$

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$\dfrac{eqtn\ 1 - eqtn2}{2\,i} \Rightarrow$

$$\sin(x) = \frac{1}{2\,i}\left(e^{ix} - e^{-ix}\right).$$

Also recall the hyperbolic trig functions

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{x}\right)$$

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right).$$

So we define, for $z \in \mathbb{C}$,

- $\cos(z) := \dfrac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \cosh(z) := \dfrac{1}{2}\left(e^{z} + e^{-z}\right) = \cos(iz)$
- $\sin(z) = \dfrac{1}{2\,i}\left(e^{iz} - e^{-iz}\right) \qquad \sinh(z) := \dfrac{1}{2}\left(e^{z} - e^{-z}\right) = -i\sin(iz).$

"Trig" identities hold, via properties of complex exponential multiplication. Note that $\sin(z)$, $\cos(z)$ are no longer bounded functions....and it's pretty challenging to figure out their transformation pictures, and their multi-valued inverse functions!

$$\sin^2 z + \cos^2 z = 1 \qquad \text{algebra.}$$

check $\quad \left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right]^2 + \left[\frac{1}{2}\left(e^{iz} + e^{-iz}\right)\right]^2 = 1$

$$\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$$

$$\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$$

trigh ...

$$\cosh^2(z) - \sinh^2(z) = \cos^2(iz) + \sin^2(iz) = 1 \quad .....$$